

# On the minimal feedback arc set of $m$ -free Digraphs \*

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## ABSTRACT

For a simple digraph  $G$ , let  $\beta(G)$  be the size of the smallest subset  $X \subseteq E(G)$  such that  $G - X$  has no directed cycles, and let  $\gamma(G)$  be the number of unordered pairs of nonadjacent vertices in  $G$ . A digraph  $G$  is called  $m$ -free if  $G$  has no directed cycles of length at most  $m$ . This paper proves that  $\beta(G) \leq \frac{1}{m-2}\gamma(G)$  for any  $m$ -free digraph  $G$ , which generalized some known results.

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## 1 Introduction

Let  $G = (V, E)$  be a digraph without loops and parallel edges, where  $V = V(G)$  is the vertex-set and  $E = E(G)$  is the edge-set.

It is well known that the cycle rank of an undirected graph  $G$  is the minimum number of edges that must be removed in order to eliminate all of the cycles in the graph. That is, if  $G$  has  $v$  vertices,  $\varepsilon$  edges, and  $\omega$  connected components, then the minimum number of edges whose deletion from  $G$  leaves an acyclic graph equals the

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cycle rank (or Betti number)  $\rho(G) = \varepsilon - v + \omega$  (see Xu [15]). However, the same problem for a digraph is quite difficult. In fact, the Betti number for a digraph was proved to be NP-complete by Karp in 1972 (see the 8th of 21 problems in [9]).

A digraph  $G$  is called to be  $m$ -free if there is no directed cycle of  $G$  with length at most  $m$ . We say  $G$  is *acyclic* if it has no directed cycles. For a digraph  $G$ , let  $\beta(G)$  be the size of the smallest subset  $X \subseteq E(G)$  such that  $G - X$  is acyclic, here  $X$  is called a *minimal feedback arc-set* of  $G$ . Let  $\gamma(G)$  be the number of unordered pairs of nonadjacent vertices in  $G$ , called the *number of missing edges* of  $G$ .

Chudnovsky, Seymour, and Sullivan [4] proved that  $\beta(G) \leq \gamma(G)$  if  $G$  is a 3-free digraph and gave the following conjecture.

**Conjecture 1.1** *If  $G$  is a 3-free digraph, then  $\beta(G) \leq \frac{1}{2}\gamma(G)$ .*

Concerning this conjecture, Dunkum, Hamburger, and Pór [5] proved that  $\beta(G) \leq 0.88\gamma(G)$ . Very recently, Chen et al. [3] improved the result to  $\beta(G) \leq 0.8616\gamma(G)$ . Conjecture 1.1 is closely related to the following special case of the conjecture proposed by Caccetta and Häggkvist [2].

**Conjecture 1.2** *Any digraph on  $n$  vertices with minimum out-degree at least  $n/3$  contains a directed triangle.*

Short of proving the conjecture, one may seek as small a value of  $c$  as possible such that every digraph on  $n$  vertices with minimum out-degree at least  $cn$  contains a triangle. This was the strategy of Caccetta and Häggkvist [2], who obtained the value  $c \leq 0.3819$ . Bondy [1] showed that  $c \leq 0.3797$ , and Shen [11] improved it to  $c \leq 0.3542$ . By using a result of Chudnovsky, Seymour and Sullivan [4] related to Conjecture 1.1, Hamburger, Haxell, and Kostochka [6] further improved this bound to 0.35312. Namely, any digraph on  $n$  vertices with minimum out-degree at least  $0.35312n$  contains a directed triangle.

More generally, Sullivan [14] proposed the following conjecture, and gave an example showing that this would be best possible if this conjecture is true. Conjecture 1.1 is the special case when  $m = 3$ .

**Conjecture 1.3** *If  $G$  is an  $m$ -free digraph with  $m \geq 3$ , then*

$$\beta(G) \leq \frac{2}{(m+1)(m-2)}\gamma(G).$$

Sullivan proved partial results of Conjecture 1.3, and showed that  $\beta(G) \leq \frac{1}{m-2}\gamma(G)$  for  $m = 4, 5$ . In this article, we prove the following theorem, which extends Sullivan's result to more general  $m$ -free digraphs for  $m \geq 4$ .

**Theorem 1.4** *If  $G$  is an  $m$ -free digraph with  $m \geq 4$ , then  $\beta(G) \leq \frac{1}{m-2}\gamma(G)$ .*

## 2 Some Lemmas

Let  $G$  be a simple digraph. For two disjoint subsets  $A, B \subseteq V(G)$ , let  $E(A, B)$  denote the set of directed edges from  $A$  to  $B$ , that is,  $E(A, B) = \{(a, b) \mid a \in A, b \in B\}$ . Let  $\bar{E}(A, B)$  be the missing edges between  $A$  and  $B$ . It follows that

$$|\bar{E}(A, B)| = |\bar{E}(B, A)| = |A| \cdot |B| - |E(A, B)| - |E(B, A)|.$$

A directed  $(v_0, v_k)$ -path  $P$  in  $G$  is a sequence of distinct vertices  $(v_0, v_1, \dots, v_{k-1}, v_k)$ , where  $(v_i, v_{i+1})$  is a directed edge for each  $i = 0, \dots, k-1$ , its length is  $k$ . Clearly, the subsequence  $(v_1, \dots, v_{k-1})$  is a  $(v_1, v_{k-1})$ -path, denoted by  $P'$ . We can denote  $P = (v_0, P', v_k)$ . A directed path  $P$  is said to be *induced* if every edge in the subgraph induced by vertices of  $P$  is contained in  $P$ .

For  $v \in V(G)$ , let  $N_i^+(v)$  be the set of vertices  $u$  such that the shortest directed  $(v, u)$ -path has length  $i$ . Similarly, let  $N_i^-(v)$  be the set of vertices whose shortest directed path to  $v$  has length  $i$ . An induced directed  $(v_0, v_k)$ -path is called to be *shortest* if  $v_k \in N_k^+(v_0)$ . From definition, we immediately have the following result.

**Lemma 2.1** *If  $(v_0, v_1, \dots, v_{k-1}, v_k)$  is a shortest induced directed  $(v_0, v_k)$ -path, then for any  $i$  and  $j$  with  $0 \leq i < j \leq k$ ,*

$$v_j \in N_{j-i}^+(v_i) \quad \text{and} \quad v_i \in N_{j-i}^-(v_j).$$

Let  $\mathcal{P}(G)$  be the set of shortest induced directed paths of  $G$ , and  $m$  be a positive integer with  $m \geq 4$ . Let  $v \in V(G)$  and  $k$  be an integer with  $1 \leq k \leq m-3$ . For any  $P \in \mathcal{P}(G)$  of length  $k-1$  and  $x, y, z \in V(G)$ , set

$$\begin{aligned} P_k(v) &= \{(x, y, z) \mid (x, P, y, z) \in \mathcal{P}(G), x = v\} \text{ and } p_k(v) = |P_k(v)|, \\ Q_k(v) &= \{(x, y, z) \mid (x, P, y, z) \in \mathcal{P}(G), y = v\} \text{ and } q_k(v) = |Q_k(v)|, \\ R_k(v) &= \{(x, y, z) \mid (x, P, y, z) \in \mathcal{P}(G), z = v\} \text{ and } r_k(v) = |R_k(v)|. \\ P'_k(v) &= \{(x, y, z) \mid (x, y, P, z) \in \mathcal{P}(G), x = v\} \text{ and } p'_k(v) = |P'_k(v)|, \\ Q'_k(v) &= \{(x, y, z) \mid (x, y, P, z) \in \mathcal{P}(G), y = v\} \text{ and } q'_k(v) = |Q'_k(v)|, \\ R'_k(v) &= \{(x, y, z) \mid (x, y, P, z) \in \mathcal{P}(G), z = v\} \text{ and } r'_k(v) = |R'_k(v)|. \end{aligned}$$

**Lemma 2.2** *For any integer  $k$  with  $1 \leq k \leq m-3$  and  $P \in \mathcal{P}(G)$  of length  $k-1$ ,*

$$\sum_{v \in V(G)} p_k(v) = \sum_{v \in V(G)} q_k(v) = \sum_{v \in V(G)} r_k(v), \quad (2.1)$$

and

$$\sum_{v \in V(G)} p'_k(v) = \sum_{v \in V(G)} q'_k(v) = \sum_{v \in V(G)} r'_k(v). \quad (2.2)$$

*Proof:* For each integer  $k$  with  $1 \leq k \leq m-3$  and  $P \in \mathcal{P}(G)$  of length  $k-1$ ,

$$\sum_{v \in V(G)} p_k(v), \quad \sum_{v \in V(G)} q_k(v), \quad \sum_{v \in V(G)} r_k(v)$$

are all equal to the number of triples  $(x, y, z)$  of distinct vertices such that  $(x, P, y, z) \in \mathcal{P}(G)$  for  $P \in \mathcal{P}(G)$ . Thus (2.1) holds. The proof of (2.2) is similar.  $\blacksquare$

**Lemma 2.3** *If  $G$  is an  $m$ -free digraph, then for any  $v \in V(G)$  and any integer  $k$  with  $1 \leq k \leq m - 3$ ,*

$$\begin{cases} p_k(v) = |E(N_{k+1}^+(v), N_{k+2}^+(v))|, \\ q_k(v) \leq |\bar{E}(N_{k+1}^-(v), N_1^+(v))|, \\ r_k(v) \leq |\bar{E}(N_1^-(v), N_{k+2}^-(v))|, \\ p'_k(v) \leq |\bar{E}(N_1^+(v), N_{k+2}^+(v))|, \\ q'_k(v) \leq |\bar{E}(N_{k+1}^+(v), N_1^-(v))|, \\ r'_k(v) = |E(N_{k+2}^-(v), N_{k+1}^-(v))|. \end{cases}$$

*Proof:* By definition, for each edge  $(u, w) \in E(N_{k+1}^+(v), N_{k+2}^+(v))$ , there exists  $v_i \in N_i^+(v)$ , for each  $i = 1, 2, \dots, k$ , such that  $(v, v_1, \dots, v_{k-1}, v_k, u, w)$  is a directed  $(v, w)$ -path of length  $k + 2$ . Since  $G$  is  $m$ -free and  $1 \leq k \leq m - 3$ , it is easy to see that  $(v, v_1, \dots, v_{k-1}, v_k, u, w)$  is a shortest induced directed path. It follows that  $(v, u, w) \in P_k(v)$  and

$$p_k(v) \geq |E(N_{k+1}^+(v), N_{k+2}^+(v))|. \quad (2.3)$$

On the other hand, for each  $(v, u, w) \in P_k(v)$ , from the definition of  $P_k(v)$  and Lemma 2.1,  $u \in N_{k+1}^+(v)$  and  $w \in N_{k+2}^+(v)$ . Thus  $(u, w) \in E(N_{k+1}^+(v), N_{k+2}^+(v))$ . It follows that

$$p_k(v) \leq |E(N_{k+1}^+(v), N_{k+2}^+(v))|. \quad (2.4)$$

Combining (2.3) and (2.4), we have that  $p_k(v) = |E(N_{k+1}^+(v), N_{k+2}^+(v))|$ . The proof of  $r'_k(v) = |E(N_{k+2}^-(v), N_{k+1}^-(v))|$  is similar.

For each  $(u, v, w) \in Q_k(v)$ , from the definition of  $Q_k(v)$  and Lemma 2.1, we have  $u \in N_{k+1}^-(v)$ ,  $w \in N_1^+(v)$  and  $uw \notin E(G)$ . Since  $G$  is  $m$ -free, we have  $(w, u) \notin E(G)$ . If not, there exists a directed cycle  $(v, w, u) \dots v$  with length  $l = k + 3 \leq m$ , a contradiction. So  $(u, w) \in |\bar{E}(N_{k+1}^-(v), N_1^+(v))|$ . Thus,  $q_k(v) \leq |\bar{E}(N_{k+1}^-(v), N_1^+(v))|$ . The proof of  $q'_k(v) \leq |\bar{E}(N_{k+1}^+(v), N_1^-(v))|$  is similar.

For each  $(u, w, v) \in R_k(v)$ , from the definition of  $R_k(v)$  and Lemma 2.1, we have  $u \in N_{k+2}^-(v)$ ,  $w \in N_1^-(v)$  and  $(u, w) \notin E(G)$ . Since  $G$  is  $m$ -free,  $(w, u) \notin E(G)$ . Otherwise, there exists a directed cycle  $(w, u, \dots, w)$  with length  $l = k + 2 \leq m - 1$ , a contradiction. Thus we have  $(u, w) \in |\bar{E}(N_1^-(v), N_{k+2}^-(v))|$ . It derives that  $r_k(v) \leq |\bar{E}(N_1^-(v), N_{k+2}^-(v))|$ . The proof of  $p'_k(v) \leq |\bar{E}(N_1^+(v), N_{k+2}^+(v))|$  is similar. ■

For any  $v \in V(G)$  and any integer  $k$  with  $1 \leq k \leq m - 3$ , set

$$\alpha_k(v) = \frac{p_k(v)}{s_k(v)} \quad \text{and} \quad \beta_k(v) = \frac{r'_k(v)}{t_k(v)}.$$

Here

$$s_k(v) = \sum_{i=k}^{m-3} p'_i(v) + \sum_{i=1}^k q'_i(v) \quad \text{and} \quad t_k(v) = \sum_{i=k}^{m-3} r_i(v) + \sum_{i=1}^k q_i(v). \quad (2.5)$$

The result is obvious.

**Lemma 2.4** *If  $a_i \geq 0, b_i \geq 0$  for each  $i = 1, 2, \dots, n$ , and  $\sum_{i=1}^n b_i > 0$ , then*

$$\min_{1 \leq i \leq n} \left\{ \frac{a_i}{b_i} \right\} \leq \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}.$$

Let

$$\alpha = \min_{\substack{v \in V(G) \\ 1 \leq k \leq m-3}} \{\alpha_k(v)\} \quad \text{and} \quad \beta = \min_{\substack{v \in V(G) \\ 1 \leq k \leq m-3}} \{\beta_k(v)\}. \quad (2.6)$$

Applying Lemma 2.4, we obtain the following bound about  $\alpha$  and  $\beta$ .

**Lemma 2.5** *If  $G$  is a  $m$ -free digraph, then*

$$\min\{\alpha, \beta\} \leq \frac{1}{m-2}.$$

*Proof:* By Lemma 2.4, we have

$$\alpha = \min_{\substack{v \in V(G) \\ 1 \leq k \leq m-3}} \{\alpha_k(v)\} = \min_{\substack{v \in V(G) \\ 1 \leq k \leq m-3}} \left\{ \frac{p_k(v)}{s_k(v)} \right\} \leq \frac{\sum_{k=1}^{m-3} \sum_{v \in V(G)} p_k(v)}{\sum_{k=1}^{m-3} \sum_{v \in V(G)} s_k(v)},$$

and

$$\beta = \min_{\substack{v \in V(G) \\ 1 \leq k \leq m-3}} \{\beta_k(v)\} = \min_{\substack{v \in V(G) \\ 1 \leq k \leq m-3}} \left\{ \frac{r'_k(v)}{t_k(v)} \right\} \leq \frac{\sum_{k=1}^{m-3} \sum_{v \in V(G)} r'_k(v)}{\sum_{k=1}^{m-3} \sum_{v \in V(G)} t_k(v)}.$$

It follows that

$$\min\{\alpha, \beta\} \leq \frac{\sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} p_k(v) + \sum_{v \in V(G)} r'_k(v) \right)}{\sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} s_k(v) + \sum_{v \in V(G)} t_k(v) \right)}. \quad (2.7)$$

Summing  $s_k(v)$  and  $t_k(v)$  over all  $v \in V(G)$  and noting (2.5), we have

$$\begin{aligned} \sum_{k=1}^{m-3} \sum_{v \in V(G)} s_k(v) &= \sum_{k=1}^{m-3} \left( \sum_{i=k}^{m-3} \sum_{v \in V(G)} p'_i(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^k \sum_{v \in V(G)} q'_i(v) \right) \\ &= \sum_{k=1}^{m-3} \left( \sum_{i=k}^{m-3} \sum_{v \in V(G)} r'_i(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^k \sum_{v \in V(G)} r'_i(v) \right) \\ &= \sum_{k=1}^{m-3} \left( \sum_{i=1}^{m-3} \sum_{v \in V(G)} r'_i(v) + \sum_{v \in V(G)} r'_k(v) \right) \\ &= (m-2) \sum_{k=1}^{m-3} \sum_{v \in V(G)} r'_k(v) \end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{m-3} \sum_{v \in V(G)} t_k(v) &= \sum_{k=1}^{m-3} \left( \sum_{i=k}^{m-3} \sum_{v \in V(G)} r_i(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^k \sum_{v \in V(G)} q_i(v) \right) \\
&= \sum_{k=1}^{m-3} \left( \sum_{i=k}^{m-3} \sum_{v \in V(G)} p_i(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^k \sum_{v \in V(G)} p_i(v) \right) \\
&= \sum_{k=1}^{m-3} \left( \sum_{i=1}^{m-3} \sum_{v \in V(G)} p_i(v) + \sum_{v \in V(G)} p_k(v) \right) \\
&= (m-2) \sum_{k=1}^{m-3} \sum_{v \in V(G)} p_k(v)
\end{aligned}$$

It follows that

$$\sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} s_k(v) + \sum_{v \in V(G)} t_k(v) \right) = (m-2) \sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} p_k(v) + \sum_{v \in V(G)} r'_k(v) \right).$$

Substituting this equality into (2.7) yields

$$\min\{\alpha, \beta\} \leq \frac{1}{m-2}.$$

The lemma follows. ■

### 3 Proof of Theorem 1.4

Clearly Theorem 1.4 holds for  $|V(G)| \leq m$ . We proceed the proof by induction on  $|V(G)|$  under the assumption that Theorem 1.4 holds for all digraphs with  $|V(G)| < n$ , here  $n > m$ . Now let  $G$  be an  $m$ -free digraph with  $|V(G)| = n$ , we may assume that for any  $v \in V(G)$ ,  $N_1^+(v) \neq \emptyset$  and  $N_1^-(v) \neq \emptyset$ . Otherwise, if there exists  $v \in V(G)$  such that  $N_1^+(v) = \emptyset$  or  $N_1^-(v) = \emptyset$ , then  $v$  is not in a directed cycle. From the inductive hypothesis, we can choose  $X \subseteq E(G-v)$  with  $|X| \leq \frac{1}{m-2} \gamma(G-v)$  such that  $(G-v) - X$  is acyclic, then  $G - X$  has no directed cycles. It follows that

$$\beta(G) \leq |X| \leq \frac{1}{m-2} \gamma(G-v) \leq \frac{1}{m-2} \gamma(G).$$

From Lemma 2.5, we have that  $\alpha \leq \frac{1}{m-2}$  or  $\beta \leq \frac{1}{m-2}$ . For each case, we prove that there exists  $X \subseteq E(G)$  satisfying  $|X| \leq \frac{1}{m-2} \gamma(G)$  and  $G - X$  has no directed cycles.

**Case 1.**  $\alpha \leq \frac{1}{m-2}$ .

By (2.6), there exist a vertex  $v \in V(G)$  and an integer  $k$  with  $1 \leq k \leq m-3$  such that

$$\alpha = \alpha_k(v) = \frac{p_k(v)}{s_k(v)} \leq \frac{1}{m-2}.$$

We consider the partition  $\{V_1, V_2\}$  of  $V(G)$ , where

$$V_1 = \bigcup_{i=1}^{k+1} N_i^+(v), \quad V_2 = V(G) \setminus V_1.$$

Clearly,  $N_1^-(v) \subset V_2$  and  $\bigcup_{i=k+2}^{m-1} N_i^+(v) \subset V_2$ . Since  $G$  is an  $m$ -free digraph, we claim

$$N_1^-(v) \cap \bigcup_{i=1}^{m-1} N_i^+(v) = \emptyset.$$

Otherwise, let  $u \in N_1^-(v) \cap \bigcup_{i=1}^{m-1} N_i^+(v)$ . Then  $(u, v) \in E(G)$  and there exists a directed  $(v, u)$ -path  $P$  with length  $l_1 \leq m - 1$ . Then  $P + (u, v)$  is a directed cycle with length  $l_1 + 1 \leq m$ , a contradiction.

Thus the number of missing edges between  $V_1$  and  $V_2$  satisfies

$$\begin{aligned} |\bar{E}(V_1, V_2)| &\geq |\bar{E}(\bigcup_{i=1}^{k+1} N_i^+(v), N_1^-(v) \cup (\bigcup_{i=k+2}^{m-1} N_i^+(v)))| \\ &\geq \sum_{i=k+2}^{m-1} |\bar{E}(N_1^+(v), N_i^+(v))| + \sum_{i=2}^{k+1} |\bar{E}(N_i^+(v), N_1^-(v))| \\ &\geq \sum_{i=k}^{m-3} p'_i(v) + \sum_{i=1}^k q'_i(v) \\ &= s_k(v). \end{aligned}$$

It follows that

$$\gamma(G) = \gamma(G_1) + \gamma(G_2) + |\bar{E}(V_1, V_2)| \geq \gamma(G_1) + \gamma(G_2) + s_k(v). \quad (3.1)$$

Let  $G_i$  be the induced subgraph by  $V_i$  for each  $i = 1, 2$ . Since  $|V_1| < n$  and  $|V_2| < n$ , from the inductive hypothesis, we have  $\beta(G_1) \leq \frac{1}{m-2}\gamma(G_1)$  and  $\beta(G_2) \leq \frac{1}{m-2}\gamma(G_2)$ . We choose  $X_i \subseteq E(G_i)$  with

$$|X_i| \leq \frac{1}{m-2}\gamma(G_i) \quad \text{for each } i = 1, 2 \quad (3.2)$$

such that  $G_i - X_i$  is acyclic.

Let  $X_3 = E(V_1, V_2)$ . Then  $X_3 = E(N_{k+1}^+(v), V_2) = E(N_{k+1}^+(v), N_{k+2}^+(v))$ , and

$$|X_3| = |E(N_{k+1}^+(v), N_{k+2}^+(v))| = p_k(v). \quad (3.3)$$

Let  $X = X_1 \cup X_2 \cup X_3$ . Then  $G - X$  has no directed cycles and, by (3.1)  $\sim$  (3.3),

$$\begin{aligned} |X| &= |X_1| + |X_2| + |X_3| \\ &= |X_1| + |X_2| + p_k(v) \\ &\leq \frac{1}{m-2}\gamma(G_1) + \frac{1}{m-2}\gamma(G_2) + \frac{1}{m-2}s_k(v) \\ &= \frac{1}{m-2}(\gamma(G_1) + \gamma(G_2) + s_k(v)) \\ &\leq \frac{1}{m-2}\gamma(G). \end{aligned}$$

**Case 2.**  $\beta \leq \frac{1}{m-2}$ .

By (2.6), there exist a vertex  $v \in V(G)$  and an integer  $k$  with  $1 \leq k \leq m-3$  such that

$$\beta = \beta_k(v) = \frac{r'_k(v)}{t_k(v)} \leq \frac{1}{m-2}.$$

We consider the partition  $\{V_1, V_2\}$  of  $V(G)$ , where

$$V_1 = \bigcup_{i=1}^{k+1} N_i^-(v), \quad V_2 = V(G) \setminus V_1.$$

Clearly,  $N_1^+(v) \subset V_2$ ,  $\bigcup_{i=k+2}^{m-1} N_i^-(v) \subset V_2$  and  $N_1^+(v) \cap \bigcup_{i=k+2}^{m-1} N_i^-(v) = \emptyset$ . The number of missing edges between  $V_1$  and  $V_2$  satisfies

$$\begin{aligned} |\bar{E}(V_1, V_2)| &\geq |\bar{E}(\bigcup_{i=1}^{k+1} N_i^-(v), N_1^+(v) \cup (\bigcup_{i=k+2}^{m-1} N_i^-(v)))| \\ &\geq \sum_{i=k+2}^{m-1} |\bar{E}(N_1^-(v), N_i^-(v))| + \sum_{i=2}^{k+1} |\bar{E}(N_i^-(v), N_1^+(v))| \\ &\geq \sum_{i=k}^{m-3} r_i(v) + \sum_{i=1}^k q_i(v) \\ &= t_k(v). \end{aligned}$$

Then

$$\gamma(G) = \gamma(G_1) + \gamma(G_2) + |\bar{E}(V_1, V_2)| \geq \gamma(G_1) + \gamma(G_2) + t_k(v).$$

Let  $G_i$  be the induced subgraph by  $V_i$  for each  $i = 1, 2$ . For  $i = 1, 2$ , from the inductive hypothesis,  $\beta(G_1) \leq \frac{1}{m-2}\gamma(G_1)$  and  $\beta(G_2) \leq \frac{1}{m-2}\gamma(G_2)$ , we can choose  $X_i \subseteq E(G_i)$  with  $|X_i| \leq \frac{1}{m-2}\gamma(G_i)$  such that  $G_i - X_i$  is acyclic. Let  $X_3 = (V_2, V_1)$ , we have  $X_3 = E(V_2, N_{k+1}^-(v)) = E(N_{k+2}^-(v), N_{k+1}^-(v))$ , and  $|X_3| = r'_k(v)$ . Let  $X = X_1 \cup X_2 \cup X_3$ . Then  $G - X$  has no directed cycles. Hence

$$\begin{aligned} |X| &= |X_1| + |X_2| + |X_3| \\ &= |X_1| + |X_2| + r'_k(v) \\ &\leq \frac{1}{m-2}\gamma(G_1) + \frac{1}{m-2}\gamma(G_2) + \frac{1}{m-2}t_k(v) \\ &= \frac{1}{m-2}(\gamma(G_1) + \gamma(G_2) + t_k(v)) \\ &\leq \frac{1}{m-2}\gamma(G). \end{aligned}$$

For each case, there exists  $X \subseteq E(G)$  satisfying  $|X| \leq \frac{1}{m-2}\gamma(G)$  and  $G - X$  has no directed cycles. This implies that  $\beta(G) \leq |X| \leq \frac{1}{m-2}\gamma(G)$ , and Theorem 1.4 follows.

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